Help Notes: derivative

30 octobre 2017

Definition 0.1. *The* **derivative of a function** f **at a number** a, *denoted by* $f'(a)$ *is*

$$f'(a) = lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

The tangent line to $y = f(x)$ *at* $(a, f(a))$ *has slope is equal to* $f'(a)$*, the derivative of* f *at* a.

Example 1 : For the function defined by $f(x) = \frac{1}{x-2}$ compute $f'(4)$ using the limit derivative definition :

Solution :

$$\begin{aligned}
&\frac{f(4+h)-f(4)}{h}\\
=&\frac{\frac{1}{4+h-2}-\frac{1}{2}}{h}\\
=&\frac{\frac{1}{2+h}-\frac{1}{2}}{h}\\
=&\frac{\frac{2-(2+h)}{2(2+h)}}{h}\\
=&\frac{\frac{-h}{2(2+h)}}{h}\\
=&\frac{-1}{2(2+h)}
\end{aligned}$$

Do not forget to simplify before passing to the limit.

$$lim_{h\to 0}\frac{-1}{2(2+h)} = \frac{-1}{4}$$

Thus the limit exist and is finite thus $f'(4) = \frac{-1}{4}$.

Example 2 : For the function defined by $f(x) = x - x^2$ compute the equation of the tangent line at $(2, -2)$ using the derivative limit definition.

Solution : Note that the domain of definition of f is $\mathbb{R}$

$$\begin{aligned}
&\frac{f(x+h)-f(x)}{h}\\
=&\frac{x+h-(x+h)^2-x+x^2}{h}\\
=&\frac{h-2hx+h^2}{h}\\
=&1-2x+h
\end{aligned}$$

$$lim_{h\to 0}1-2x+h=1-2x$$

Thus the limit exist and is finite thus $f'(x)=1-2x$.

Be careful to answer fully answer the question that is asked, we need the equation of the tangent here.

Moreover, we know that the slope of the tangent at $(2,-2)$ is $m=f'(2)=-3$. The equation of the tangent at $(2,-2)$ is $y=-3x+p$ where $p\in\mathbb{R}$. Since $(2,-2)$ belongs to the tangent we have $-2=-3\times 2+p=-6+p$ and $p=4$. Finally, the equation of the tangent is $y=-3x+4$.

Definition 0.2. *A function f is* **differentiable at a** *if $f'(a)$ exists. It is* **differentiable on a open interval** (a,b) **(or** (a,∞) **or** $(-\infty,a)$ **or** $(-\infty,\infty)$**)** *if it is differentiable at every number in the interval. If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')'=f''$. This new function is called the* **second derivative of f.** *The* **third derivative** *is the derivative of the second derivative denoted b y $(f'')'=f'''$. The process continues. In general the n^{th}* **derivative** *is denoted by $f^{(n)}$ and is obtained by differentiating f n^{th} time.*

Example : Let $f(x)=\sqrt{x-4}$, find the general derivative $f'(x)$ as a new function by using the derivative definition.

Solution : Note that the domain of definition of f is $[4,\infty)$.

$$\begin{aligned}&\frac{f(x+h)-f(x)}{h}\\ =&\frac{\sqrt{x-4+h}-\sqrt{x-4}}{h}\\ =&\frac{h}{h(\sqrt{x-4+h}+\sqrt{x-4})}\\ =&\frac{1}{\sqrt{x-4+h}+\sqrt{x-4}}\end{aligned}$$

For $x\neq 4$

$$lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x-4}}$$

Thus the limit exist and is finite thus $f'(x)=\frac{1}{2\sqrt{x-4}}$. For $x=4$,

$$\frac{f(x+h)-f(x)}{h}=\frac{1}{\sqrt{h}}$$

But then $lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\infty$ as a consequence $f'(4)$ does not exist.

As a conclusion, f is differentiable over $(4,\infty)$ and

$$f'(x)=\frac{1}{2\sqrt{x-4}}$$

Advise : When using derivability condition, first compute $\frac{f(x+h)-f(x)}{h}$ until you get a for that permits you to decide about the limit when $h\to 0$, if this limit exists and is finite then the derivative is whatever you found as a limit,

$$f'(x) = lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if it does not exist or is not finite, then the derivative does not exist for those point, and f is not differentiable at those points.

Theorem 0.3. *If f is differentiable at a then f is continuous at a. (BE AWARE : The converse is wrong).*

Definition 0.4. *If we have a parametric curve defined with $x = f(t)$ and $y = g(t)$, the slope of tangent at the point (x, y) is*

$$\frac{dy}{dx} = \frac{g'(t_0)}{f'(t_0)} = \frac{\frac{dg(t_0)}{dt}}{\frac{df(t_0)}{dt}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

where t_0 such that $x = f(t_0)$ and $y = g(t_0)$.

Example : Find an equation of the tangent line to the parametric curve

$$x = 2sin(2t) \ y = 2sin(t)$$

at the point $(\sqrt{3}, 1/2)$. Where does this curve has horizontal or vertical tangent ?

Solution : At the point with parameter value t, the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2sin(t))}{\frac{d}{dt}(2sin(2t))} = \frac{2cos(t)}{2(cos(2t)(2)} = \frac{cos(t)}{2cos(2t)}.$$

The point $(\sqrt{3}, 1)$ corresponds to the parameter value $t = \pi/6$ since when $t = \pi/6$ we have $2sin(2t) = \sqrt{3}$ and $sin(t) = 1$, so the slope of the tangent at that point is

$$\frac{dy}{dx}\Big|_{t=\pi/6} = \frac{cos(\pi/6)}{2cos(\pi/3)} = \frac{\sqrt{3}/2}{2(1/2)} = \frac{\sqrt{3}}{2}$$

An equation of the tangent line is therefore

$$y - 1 = \frac{\sqrt{3}}{2}(x - \sqrt{3}) \ \ or \ \ y = \frac{\sqrt{3}}{2}x - 1/2$$

Implicit Differentiation

If y is defined implicitly as a function of x by an equation relating x and y, we treat y as a differentiable function of x and proceed as follows :

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with y' (or $\frac{dy}{dx}$) on one side of the equation and solve for y'.

Example 1 : Find $y'' = d^2y/d^2x$ using implicit differentiation when $2x^2 - 3y^2 = 4$

Solution : BE CAREFUL : DIFFERENTIATE BOTH SIDE OF THE EQUALITY !!!!!!

We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get :

$$4x - 3y'y = 0$$

That is

$$y' = \frac{4x}{3y}$$

Then, applying implicit differentiation again as before to the last equality we obtain :

$$y'' = \frac{12y - 12y'}{9y^2} = \frac{12y - 12\frac{4x}{3y}}{9y^2} = \frac{4y - 4\frac{4x}{3y}}{3y^2} = \frac{12y^2 - 16x}{3y^3}$$

Example 2 : Use implicit differentiation to find the slope of the tangent line to the curve at the point $(1, \sqrt[4]{15})$ for $x^4 + y^4 = 16$;

Solution : The slope of the tangent at a point with abscissa x is equal to the derivative of the function at this point. So in order to do this exercise we will consider y to be a function of x and do an implicit differentiation, so that we can find the differentiation of y at the point x and then compute y' for x at the point specified.

BE CAREFUL : DIFFERENTIATE BOTH SIDE OF THE EQUALITY !!!!!!

We use implicit differentiation on the previous equation. We differentiate with respect to x thinking of y as a function of x, we get :

$$4x^3 + 4y'y^3 = 0$$

Thus we obtain

$$y' = \frac{-4x^3}{4y^3} = \frac{-x^3}{y^3}$$

Then at the point $(1, \sqrt[4]{15})$ we obtain

$$y'|_{(1, \sqrt[4]{15})} = \frac{-1}{\sqrt[4]{15}^3}$$

Thus the slope of the tangent at the point $(1, \sqrt[4]{15})$ is $\frac{-1}{\sqrt[4]{15}^3}$.

Steps in logarithm differentiation Differentiate

1. Take natural logarithm in both side of the equality and use logarithm laws.
2. Differentiate implicitly with respect to x
3. Solve the resulting equation.

Example : $f(x) = ln\left(\frac{x^5}{(2x-1)^3(x^2+1)}\right)$

Solution : Note that using the property of the logarithm we have that

$$f(x) = ln\big(\frac{x^5}{(2x-1)^3(x^2+1)}\big) = 5ln(x) - 3ln(2x-1) - ln(x^2+1)$$

Thus, applying two chain rules, we get

$$f'(x) = \frac{5}{x} - 3\frac{2x}{2x-1} - \frac{2x}{x^2+1} = \frac{5}{x} - \frac{6x}{2x-1} - \frac{2x}{x^2+1}$$

Linear approximation

The approximation

$$f(x) \sim f(a) + f'(a)(x-a)$$

is called the **linear approximation** or **tangent line approximation of** f **at** a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x-a)$$

is called the **linearization** of f at a.

Function	**Derivative**	
c	0	
x	1	
$x^n, \ n \in \mathbb{R}$	nx^{n-1}	
e^x	e^x	
$cf(x)$	$cf'(x)$	
$f(x)+g(x)$	$f'(x)+g'(x)$	
$f(x)-g(x)$	$f'(x)-g'(x)$	
$f(x)g(x)$	$f(x)g'(x)+f'(x)g(x)$	Product rule
$\frac{f(x)}{g(x)}$	$\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$	Quotient rule
$f(g(x))$	$g'(x)f'(g(x))$	Chain rule
$u(x)^n$	$nu'(x)u(x)^{n-1}$	
a^x	$a^x ln(a)$	
a^b	0	
$(f(x))^b$	$bf'(x)f(x)^{b-1}$	
$a^{g(x)}, \ a>0$	$a^{g(x)}ln(a)g'(x)$	
$log_a(x), \ a>0$	$\frac{1}{xln(a)}$	
$ln(x)$	$1/x$	
$ln(u(x))$	$\frac{u'(x)}{u(x)}$	
$ln\lvert x\rvert$	$1/x$	
$\frac{1}{g(x)}$	$-\frac{g'(x)}{g(x)^2}$	
$cos(x)$	$-sin(x)$	
$sin(x)$	$cos(x)$	
$tan(x)=\frac{sin(x)}{cos(x)}$	$sec^2(x)=\frac{1}{cos^2(x)}$	
$sec(x)=\frac{1}{cos(x)}$	$sec(x)tan(x)$	
$csc(x)=\frac{1}{sin(x)}$	$-csc(x)cot(x)$	
$cot(x)=\frac{1}{tan(x)}$	$-csc^2(x)$	
$sin^{-1}(x)=arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$	
$cos^{-1}(x)=arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$	
$tan^{-1}(x)=arctan(x)$	$\frac{1}{1+x^2}$	
$sinh(x)=\frac{e^x-e^{-x}}{2}$	$cosh(x)$	
$cosh(x)=\frac{e^x+e^{-x}}{2}$	$sinh(x)$	
$tanh(x)=\frac{sinh(x)}{cosh(x)}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$	$sech^2(x)$	
$coth(x)=\frac{cosh(x)}{sinh(x)}=\frac{e^x+e^{-x}}{e^x-e^{-x}}$	$-csh^2(x)$	
$csch(x)=\frac{1}{sinh(x)}=\frac{2}{e^x-e^{-x}}$	$-sech(x)tanh(x)$	
$sech(x)=\frac{1}{sinh(x)}=\frac{2}{e^x+e^{-x}}$	$-csch(x)coth(x)$	
$sinh^{-1}(x)=ln(x+\sqrt{x^2+1})$	$\frac{1}{\sqrt{x^2+1}}$	
$cosh^{-1}(x)=ln(x+\sqrt{x^2+1})$	$\frac{1}{\sqrt{x^2-1}}$	
$tanh^{-1}(x)=1/2ln\left(\frac{1+x}{1-x}\right)$	$\frac{1}{1-x^2}$	
$sech^{-1}(x)=ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$\frac{1}{x\sqrt{1-x^2}}$	

About finding derivative using the previous table

It is very important that you **take time observing your function.** See if it is a good idea to rewrite it before taking the derivative (if there is not a way to rewrite it to make the computation easier and avoid computational mistakes). Then try to anticipate which rule you will need to use (product, chain, quotient...). If necessary split the derivative in various step especially if you need to use various rules. Tell me which rules you will apply.

Example : Compute the derivative of the following function :

1. $f(x) = \frac{3x-5x^3}{\sqrt[3]{x}}$
   Solution
   Note that
   $$f(x) = 2x^{2/3} - 5x^{9/3},$$
   thus using the power rule we get
   $$f'(x) = 4/3x^{-1/3} - \frac{40}{3}x^{8/3}$$
2. $f(z) = \sqrt{5}z + \sqrt{11z}$
   Solution
   Note that
   $$f(z) = \sqrt{5}z + \sqrt{11z} = \sqrt{5}z + \sqrt{11}\sqrt{z}$$
   Thus, using the power rule we get :
   $$f'(z) = \sqrt{5} + \sqrt{11}/2z^{-1/2}$$
3. $f(x) = (1 + 2x + 3x^2)(5x^5 - 4x^4)$
   Solution
   We write $f(x) = u(x)v(x)$, with $u(x) = 1 + 2x + 3x^2$ thus $u'(x) = 6x + 2$ and $v(x) = 5x^5 - 4x^4 = x^4(5x - 4)$ thus $v'(x) = 25x^4 - 16x^3 = x^3(25x - 16)$.
   Using the product rule we get :
   $$\begin{aligned} f'(x) &= u'(x)v(x) + v'(x)u(x) = x^4(5x-4)(6x+2) + x^3(25x-16)(1+2x+3x^2) \\ &= x^3(x(5x-4)(6x+2) + (25x-16)(1+2x+3x^2)) \\ &= x^3(105x^3 - 12x^2 - 15x - 16) \end{aligned}$$
4. $f(x) = \frac{x^3+1}{3-x}$
   Solution :
   Note that
   $$f(x) = \frac{u(x)}{v(x)}$$
   where $u(x) = x^3 + 1$ thus $u'(x) = 3x^2$ and $v(x) = 3 - x$ thus $v'(x) = -1$.
   Then, using the quotient rule we get
   $$f'(x) = \frac{u'(x)v(x) - v'(x)u(x)}{v^2(x)} = \frac{3x^2(3-x) + (x^3+1)}{(3-x)^2} = \frac{-2x^3 + 9x^2 + 1}{(3-x)^2}$$

5. $f(x) = \frac{\sqrt{x^3}}{sec(x)(2x+3)}$
f is a quotient of a square root function by the product of a trigonometric function and a polynomial function. .
Then, using the quotient and product rules we get :

$$f'(x) = \frac{3/2\sqrt{x}sec(x)(2x+3) - \sqrt{x^3}(sec(x)tan(x)(2x+3) + 2sec(x))}{(sec(x)(2x+3))^2}$$

6. $f(x) = (1 + (2x+1)^6)^7$
Solution : f is a polynomial function thus defined and differentiable for any real values.
Thus using the chain rule twice we get :

$$f'(x) = 84(2x+1)^5(1 + (2x+1)^6)^6$$

7. $f(x) = \sqrt{cos(x^2 - 3x^{-8} + 1)}$
Solution : f is the square root of a composite of a trigonometric with a polynomial function.
Here I have to deal with a composite of a composite too much to think about in my head at once thus I prefer to be careful again.
Note that $f(x) = u(v(x))$ where $u(x) = \sqrt{x}$ thus $u'(x) = \frac{1}{2\sqrt{x}}$ and $v(x) = cos(x^2 - 3x^{-8} + 1)$ thus using the chain rule, $v'(x) = -(2x - 24x^{-9})sin(x^2 - 3x^{-8} + 1)$.
Thus using the chain rule twice, we get

$$f'(x) = v'(x)u'(v(x)) = -(2x - 24x^{-9})sin(x^2 - 3x^{-8} + 1)\frac{1}{2\sqrt{cos(x^2 - 3x^{-8} + 1)}}$$

8. $f(z) = \frac{2^z - z^2}{1 - log_3(z)}$
Solution : f is the quotient of a polynomial with a log function.
Remember $2^z = e^{ln(2)z}$ **! !**
Using the quotient, exponential and logarithm rules we get :

$$f'(x) = \frac{(ln(2)2^z - 2z)(1 - log_3(z)) - \frac{2^z - z^2}{ln(3)z}}{(1 - log_3(z))^2}$$

9. $f(x) = \frac{3^{-sin(x)}}{1 + ln(x^3 - x)}$
Solution : f is a quotient of a composite of an exponential with a trigonometric function and sum of a constant with a composite of a logarithm with a polynomial function.
Using the chain rule twice and the quotient rule we get

$$\begin{aligned} f'(x) &= \frac{-ln(3)cos(x)3^{-sin(x)}(1+ln(x^3-x)) + 3^{-sin(x)}\frac{3x^2-1}{x^3-x}}{(1+ln(x^3-x))^2} \\ &= \frac{3^{-sin(x)}(-ln(3)cos(x)(1+ln(x^3-x)) + \frac{3x^2-1}{x^3-x})}{(1+ln(x^3-x))^2} \end{aligned}$$